Massive neutrinos, massless neutrinos, and $so(4,2)$ invariance.

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Abstract

Dirac's equation for a massless particle is conformal invariant, and accordingly has an $so(4,2)$ invariance algebra. It is known that although Dirac's equation for a massive spin 1/2 particle is not conformal invariant, it too has an $so(4,2)$ invariance algebra. It is shown here that the algebra of operators associated with a 4-component massless particle, or two flavors of 2-component massless particles, can be deformed into the algebra of operators associated with a spin 1/2 particle with positive rest mass. It is speculated that this may be exploited to describe massless neutrino mixing.

1 Introduction

In the 1970s and 1980s, Asim Barut's name became synonymous with the applications to quantum mechanics of the Lie algebra $so(4,2)$, whether as a space-time (conformal) symmetry algebra [1], as a spectrum generating algebra in generalised Coulomb problems [2], or as an algebra associated with Dirac's equation for the electron [3]. I would like to revisit from that period an idea involving $so(4,2)$ that may have renewed relevance because of the subsequent discovery of several neutrino types, some of which may have small rest masses.

It is well known that the massless Dirac equation is not only Poincaré invariant but also $so(4,2)$ (conformal) invariant [4]. More surprising is that the massive Dirac equation also has an $so(4,2)$ invariance algebra [5]. The solution spaces of these two equations each carry hermitian representations of that Lie algebra. However, only in the massless case can this algebra be interpreted as the Lie algebra of the conformal group. In the massive case, the $so(4,2)$ invariance algebra contains the physical homogeneous Lorentz subalgebra, but not the generators of the physical translations, dilatations or special conformal transformations.

What was done in Ref.[5] was to construct, from the operators spanning the usual hermitian representation of the Poincaré Lie algebra on the positive-energy solution space of the *massive* Dirac equation, a set of operators that span another, inequivalent hermitian irreducible representation of the Poincaré Lie algebra – and of its extension, the Lie algebra $so(4,2)$ – of the type appropriate for the description of a *massless particle* with a definite helicity. What will be shown below is that if two types of four-component neutrinos are considered, and if a parameter with the dimensions of mass is introduced, then we can construct from the algebra of operators for the massless particles, a set of operators that span a representation of the Poincaré Lie algebra appropriate to the description of a spin 1/2 particle with nonzero rest mass. In short, the operator algebra for the set of massless particles can be deformed into the operator algebra of a massive particle, acting on the same vector space.

The observed mixing of the three known neutrino types is thought to be

inexplicable unless some types have nonzero rest masses [6]. Neutrino masses have not been measured directly, but it is believed that if they are nonzero, then they are very small compared with the rest mass of the electron. This gives renewed relevance to the the notions of conformal invariance in the massless case, and $so(4,2)$ invariance in the massive case, and the relationship between the two.

2 Massless and massive Dirac equations, and $so(4,2)$ invariance.

The space of positive energy solutions of the massless Dirac equation

$$(\gamma \cdot P)\,\psi(x) \equiv \gamma_\mu P^\mu\,\psi(x) = 0\,, \qquad P_\mu = i\partial/\partial x^\mu\,, \tag{1}$$

when restricted by the helicity condition

$$\gamma_5\,\psi(x) = \pm\,\psi(x)\,, \tag{2}$$

carries a unitary representation of the Poincaré group appropriate to the description of a massless particle with helicity $\mp 1/2$. Here $x = (x^\mu)$, $\mu = 0,\ 1,\ 2,\ 3$, and the Dirac matrices γ^μ satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \quad \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\,. \tag{3}$$

We choose the diagonal metric tensor with $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$, and set both $\hbar$ and c equal to 1.

This representation of the Poincaré group extends to a unitary representation of the conformal group, with generators $M_{AB} = -M_{BA}$, $A, B = 0,\ 1,\ 2,\ 3,\ 5,\ 6,$ given by [1, 4]

$$M_{\mu\nu} = J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \frac{1}{4}i[\gamma_\mu, \gamma_\nu]\,,$$

$$M_{56} = D = x \cdot P + \frac{3}{2}i\,, \quad M_{\mu 6} - M_{\mu 5} = P_\mu\,,$$

$$M_{\mu 6} + M_{\mu 5} = K_\mu = x_\mu(2D + i) - (x \cdot x)\,P_\mu - i(\gamma \cdot x)\gamma_\mu\,. \tag{4}$$

These satisfy the commutation relations

$$[M_{AB}, M_{CD}] =$$

$$-i\left(g_{AC}M_{BD} + g_{BD}M_{AC} - g_{AD}M_{BC} - g_{BC}M_{AD}\right)\,, \tag{5}$$

where the extended metric tensor is diagonal with $g_{55} = -1$, $g_{66} = 1$. It is at once clear from (5) and the form of this diagonal metric that the Lie algebra of the conformal group is isomorphic to $so(4,2)$.

We consider a 4-component massless particle, described by (1) but now without the fixed helicity condition (2). Noting that the generators (4) are not all dimensionless, we also introduce a parameter $\mathbb{M} > 0$ with the dimensions of a mass (or an inverse length, since we have set $\hbar = c = 1$). Then we can modify the conformal algebra (4) to give a new set of dimensionless operators N_{AB} that satisfy the same $so(4,2)$ relations (5) as the M_{AB}, namely

$$N_{\mu\nu} = M_{\mu\nu} = J_{\mu\nu}\,, \quad N_{56} = \gamma_5 D\,,$$

$$N_{\mu 5} = \frac{1}{2}\big(\mathbb{M}K_\mu - P_\mu/\mathbb{M}\big)\,, \quad N_{\mu 6} = \frac{1}{2}\gamma_5\big(\mathbb{M}K_\mu + P_\mu/\mathbb{M}\big)\,. \tag{6}$$

Our motivation for the change from the M_{AB} to the N_{AB}, as will become clearer below, is to make the operator structure in the massless case mirror that for the massive Dirac equation

$$(\gamma\cdot P)\,\psi(x) = m\,\psi(x)\,, \tag{7}$$

for which the corresponding $so(4,2)$ invariance algebra is spanned by the operators T_{AB} defined by [5]

$$T_{\mu\nu} = J_{\mu\nu}\,, \quad T_{56} = J = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}J_{\mu\nu}J_{\rho\sigma}\,,$$

$$T_{\mu 5} = \frac{1}{2m}\left\{J_{\mu\nu}, P^\nu\right\}\,, \quad T_{\mu 6} = \frac{1}{2m}\left\{P_\mu, J\right\}\,. \tag{8}$$

Here $\{\,,\,\}$ denotes the anticommutator, and $\epsilon^{\mu\nu\rho\sigma}$ is the alternating tensor with $\epsilon^{0123} = 1$. Once again, the operators T_{AB} satisfy relations of the form (5).

The key to understanding the rather surpising $so(4,2)$ invariance of the massive Dirac equation is that, when regarded as *reducible* representations of the *homogeneous* Lorentz Lie algebra $so(3,1)$, two irreducible representations of the Poincaré Lie algebra are equivalent[8]: the one appropriate to the description of a positive energy massless particle with definite helicity, equal to either $+1/2$ or $-1/2$; and the other appropriate to the description of a positive energy massive particle with spin 1/2. Each of these reducible representations of $so(3,1)$ extends to an hermitian representation of $so(4,2)$. It is for this reason that the $J_{\mu\nu}$, as generators of homogeneous Lorentz transformations for the massless particle, can be identified with the $N_{\mu\nu}$ as in (6); and similarly that as generators of homogeneous Lorentz transformations for the massive particle, they can be identified with the $T_{\mu\nu}$ as in (8).

It is known that the operators (4) satisfy certain characteristic 'representation relations' [9], and the same is true of the operators (6) and (8). More generally, and more importantly, the similar structures of the algebras of operators associated with the massless particles and with the massive particle enable us now to proceed towards our goal.

3 Deformation of the massless algebra to the massive algebra.

Our problem is to construct, on the space of solutions of the 4-component massless Dirac equation (1), a set of operators $\mathbb{P}_\mu$, $\mathbb{J}_{\mu\nu}$ that satisfy the defining relations of a representation of the Poincaré Lie algebra, appropriate for the description of a spin 1/2 particle with rest mass $\mathbb{M}$. We do not deform the homogeneous Lorentz subalgebra, and take $\mathbb{J}_{\mu\nu} = J_{\mu\nu}$ as in (4).

In order to proceed with the construction of $\mathbb{P}_\mu$, we first see how the operator P_μ is embedded in the operator algebra associated with the massive Dirac equation (7, 8). Although $T_{\mu 5}$ and $T_{\mu 6}$ are expressed in terms of the P_μ and $J_{\mu\nu}$ in (8), it is not possible to invert these relations and express P_μ in terms of the $J_{\mu\nu}$, $T_{\mu 5}$ and $T_{\mu 6}$ alone. There are three independent 4-vectors acting on the space of solutions to (7). They could be taken to be P_μ, $J_{\mu\nu}P^\nu$ and

$J_{\mu\nu} J^{\nu\rho} P_\rho$ but a more convenient set for our purposes is

$$\begin{aligned}
T_{\mu 6} + T_{\mu 5} &= \frac{1}{2m}\{J, P_\mu\} + \frac{1}{2m}\{J_{\mu\nu}, P^\nu\}\,, \\
&= \frac{1}{m}\big((J - 3i/2)P_\mu + iW_\mu + J_{\mu\nu}P^\nu\big)\,, \\
T_{\mu 6} - T_{\mu 5} &= \frac{1}{2m}\{J, P_\mu\} - \frac{1}{2m}\{J_{\mu\nu}, P^\nu\}\,, \\
&= \frac{1}{m}\big((J + 3i/2)P_\mu + iW_\mu + J_{\mu\nu}P^\nu\big)\,, \\
Z_\mu &= \frac{1}{m}\left(-iJW_\mu - \frac{i}{2}J_{\mu\nu}P^\nu - \frac{1}{2}P_\mu\right)\,, \qquad (9)
\end{aligned}$$

where $W_\mu = (1/2)\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma$ is the Pauli-Lubanski 4-vector, and $J = T_{56}$ is as in (8).

The first two of these 4-vector operators satisfy

$$\begin{aligned}
J\,(T_{\mu 6} + T_{\mu 5}) &= (T_{\mu 6} + T_{\mu 5})\,(J + i)\,, \\
J\,(T_{\mu 6} - T_{\mu 5}) &= (T_{\mu 6} - T_{\mu 5})\,(J - i)\,, \qquad (10)
\end{aligned}$$

which are among the relations of the form (5) satisfied by the T_{AB}, and it is straightforward to check that, in addition,

$$J\,Z_\mu = -Z_\mu\,J\,. \qquad (11)$$

Operators satisfying similar shifting relations to J, $T_{\mu 6} + T_{\mu 5}$ and $T_{\mu 6} - T_{\mu 5}$ are easily found in the massless case, namely $N_{56} = \gamma_5 D$, $N_{\mu 6} + N_{\mu 5}$ and $N_{\mu 6} - N_{\mu 5}$ with, again as a consequence of the $so(4,2)$ commutation relations of the form (5) satisfied by the N_{AB},

$$\begin{aligned}
(\gamma_5 D)\,(N_{\mu 6} + N_{\mu 5}) &= (N_{\mu 6} + N_{\mu 5})\,(\gamma_5 D + i)\,, \\
(\gamma_5 D)\,(N_{\mu 6} - N_{\mu 5}) &= (N_{\mu 6} - N_{\mu 5})\,(\gamma_5 D - i)\,. \qquad (12)
\end{aligned}$$

To find an analogue of Z_μ, we note that on solutions of (7),

$$Z_\mu = -\frac{i}{2}[(\gamma \cdot x)P_\mu - (D - i/2)\gamma_\mu - m(\gamma \cdot x)\gamma_\mu + m x_\mu] \,, \tag{13}$$

where D is as in (4). This suggest the choice

$$\zeta_\mu = -\frac{i}{2}\tau_2[(\gamma \cdot x)P_\mu - (D - i/2)\gamma_\mu] \,, \tag{14}$$

and it is then easily checked that indeed

$$(\gamma_5 D)\,\zeta_\mu = -\zeta_\mu\,(\gamma_5 D) \,, \tag{15}$$

analogous to equation (11). Furthermore, it is also easily checked that

$$(\gamma \cdot P)\zeta_\mu = -\zeta_\mu(\gamma \cdot P) \,, \tag{16}$$

so that ζ_μ leaves invariant the space of solutions of the massless Dirac equation (1), as do the N_{AB}. Note however that ζ_μ anticommutes with γ_5. It is for this reason that we have dropped the condition (2).

The relations (9) can be inverted to give, on solutions of the massive Dirac equation (7),

$$P_\mu = \frac{m}{2}[J^2 + \frac{1}{4}]^{-1}$$

$$\times\big((J + i/2)(T_{\mu 6} + T_{\mu 5}) + (J - i/2)(T_{\mu 6} - T_{\mu 5}) + 2Z_\mu\big) \,. \tag{17}$$

This suggests a formula for $\mathbb{P}_\mu$ on solutions of the massless Dirac equation (1), namely

$$\mathbb{P}_\mu = \frac{\mathbb{M}}{2}(D^2 + \frac{1}{4})^{-1}$$

$$\times\big((\gamma_5 D + i/2)(N_{\mu 6} + N_{\mu 5}) + (\gamma_5 D - i/2)(N_{\mu 6} - N_{\mu 5}) + 2\zeta_\mu\big) \tag{18}$$

which simplifies to

$$\mathbb{P}_\mu = \frac{1}{2}(D^2 + \frac{1}{4})^{-1}$$

$$\times\big(\mathbb{M}^2(D + i/2)K_\mu + (D - i/2)P_\mu + 2\mathbb{M}\zeta_\mu\big) \,. \tag{19}$$

Note that $[J^2 + 1/4]^{-1}$ and $[D^2 + 1/4]^{-1}$ are well-defined, nonsingular operators, and that $\mathbb{P}_\mu$ has the appropriate dimensions.

It is now possible to check by direct calculation that the operators $\mathbb{P}_\mu$ and $\mathbb{J}_{\mu\nu} = J_{\mu\nu}$ satisfy all the algebraic relations appropriate for the description of a spin 1/2 massive particle of rest mass $\mathbb{M}$.

To check that

$$[\mathbb{P}_\mu, \mathbb{P}_\nu] = 0 \,, \tag{20}$$

we introduce the notation

$$A_\mu B_\nu - A_\nu B_\mu = A_{[\mu} B_{\nu]} \,, \tag{21}$$

so that in particular $[\mathbb{P}_\mu, \mathbb{P}_\nu] = \mathbb{P}_{[\mu}\mathbb{P}_{\nu]}$. Then we find that, on the solution space of (1),

$$K_{[\mu} P_{\nu]} = 2(D - i/2) L_{\mu\nu} - i R_{\mu\nu} \,,$$

$$P_{[\mu} K_{\nu]} = -2(D + 3i/2) L_{\mu\nu} + i R_{\mu\nu} - 4i S_{\mu\nu} \,,$$

$$\zeta_{[\mu} \zeta_{\nu]} = -2(D - i/2) L_{\mu\nu} + 2D R_{\mu\nu} - 4i(D - i/2)^2 S_{\mu\nu} \,, \tag{22}$$

where

$$L_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu \,,$$

$$R_{\mu\nu} = (\gamma \cdot x)(\gamma_\mu P_\nu - \gamma_\nu P_\mu) \,,$$

$$S_{\mu\nu} = (i/4)[\gamma_\mu, \gamma_\nu] \,. \tag{23}$$

Furthermore, again on the solutions of (1), we find that

$$(D + i/2) K_{[\mu} \zeta_{\nu]} = -(D - 3i/2)) \zeta_{[\mu} K_{\nu]} \,,$$

$$(D - i/2) P_{[\mu} \zeta_{\nu]} = -(D + 3i/2) \zeta_{[\mu} P_{\nu]} \,. \tag{24}$$

The relations (22) and (24), together with (19) and the shifting formulas (12, 15), lead to the desired result (20).

Similarly, to check that (19) leads to

$$\mathbb{P} \cdot \mathbb{P} = \mathbb{M}^2 \,, \tag{25}$$

we use the results on solutions of (1)

$$\begin{aligned} K \cdot P &= 2(D - i/2)(D - 3i/2) \,, \\ P \cdot K &= 2(D + i/2)(D + 3i/2) \,, \end{aligned} \tag{26}$$

and

$$P \cdot \zeta = \zeta \cdot P = 0 \,, \quad K \cdot \zeta = \zeta \cdot K = 0 \,, \quad \zeta \cdot \zeta = 2D^2 + 1/2 \,. \tag{27}$$

Several other useful formulas hold on the solutions of (1), namely

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} K_\sigma = -\gamma_5 K^\mu \,, \qquad & \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma = \gamma_5 P^\mu \,, \\ \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} \zeta_\sigma \;\; &= \;\; 2i\gamma_5 D \zeta^\mu \,, \end{aligned} \tag{28}$$

$$\begin{aligned} \{\gamma_5 D, K_\mu\} = 2\gamma_5(D - i/2)K_\mu \,, \qquad & \{\gamma_5 D, \zeta_\mu\} = 0 \,, \\ \{\gamma_5 D, P_\mu\} \;\; &= \;\; 2\gamma_5(D + i/2)P_\mu \,, \end{aligned} \tag{29}$$

and

$$\begin{aligned} \{J_{\mu\nu}, K^\nu\} = 2(D - i/2)K_\mu \,, \qquad & \{J_{\mu\nu}, \zeta^\nu\} = 0 \,. \\ \{J_{\mu\nu}, P^\nu\} \;\; &= \;\; 2(D + i/2)P_\mu \,, \end{aligned} \tag{30}$$

To obtain the Pauli-Lubanski 4-vector associated with $\mathbb{P}_\mu$ and $\mathbb{J}_{\mu\nu}$, we use the results (28) to deduce from (19) that

$$\begin{aligned}
\mathbb{W}^\mu &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathbb{J}_{\nu\rho}\mathbb{P}_\sigma = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}\mathbb{P}_\sigma \\
&= -\frac{1}{4}(D^2+1/4)^{-1}\gamma_5\left(\mathbb{M}^2(D+i/2)K^\mu\right. \\
&\quad \left. -(D-i/2)P^\mu - 4i\mathbb{M}D\zeta^\mu\right) . \qquad (31)
\end{aligned}$$

Then with the help of (26, 27) we get

$$\mathbb{W}\cdot\mathbb{W} = -\frac{1}{2}(\frac{1}{2}+1)\mathbb{M}^2 , \qquad (32)$$

as required for a massive particle with spin 1/2.

Finally, we note with the help of (29, 30) that in terms of $\mathbb{P}_\mu$ and $\mathbb{J}_{\mu\nu}$, the $so(4,2)$ generators (6) in the massless case can be rewritten as

$$\begin{gathered}
N_{\mu\nu} = \mathbb{J}_{\mu\nu} , \quad N_{56} = \gamma_5 D = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\mathbb{J}_{\mu\nu}\mathbb{J}_{\rho\sigma} , \\
N_{\mu 5} = \frac{1}{2\mathbb{M}}\{\mathbb{J}_{\mu\nu},\mathbb{P}^\nu\} , \quad N_{\mu 6} = \frac{1}{2\mathbb{M}}\{\mathbb{P}_\mu,\gamma_5 D\} . \qquad (33)
\end{gathered}$$

which have the same form as the formulas (8) in the massive case.

4 Concluding remarks.

The algebra of operators associated with a 4-component massless particle has been deformed to obtain the operator algebra associated with a spin 1/2 particle with positive rest mass.

Instead of dropping the helicity condition (2), we could instead have introduced two flavors of 2-component massless particles, and an associated set of Pauli flavor matrices τ_1, τ_2, τ_3 that commute with all Dirac matrices. Then we could have replaced γ_5 in (2) by $\tau_3\gamma_5$, and our calculations would go through as before provided that we replaced ζ_μ everywhere, and in particular in (19) and (31), by $\tau_1\zeta_\mu$. The operators $\mathbb{P}_\mu$ and $\mathbb{W}_\mu$ so modified leave

invariant the space of solutions of (1) and the modified helicity condition. In this way, the operator algebra associated with two flavors of 2-component massless particles can also be deformed to obtain the algebra of a single spin 1/2 particle with positive rest mass.

From a geometric point of view it is notable that non-null 4-vector operators $\mathbb{P}_\mu$ and $\mathbb{W}_\mu$ can be constructed on the state space of a set of massless particles, for which the 4-vector operators P_μ and K_μ are null.

It is hoped that these results can be exploited to describe the mixing of neutrino types without the introduction of neutrino rest masses.

References

[1] Barut, A.O. and Brittin, W.E., *De Sitter and Conformal Groups and their Applications*, Lecture Notes in Theoretical Physics Vol. XIII (Colorado Associated University Press, Boulder, 1971).

[2] Barut, A.O., *Dynamical Groups and Generalized Symmetries in Quantum Theory*, (University of Canterbury, Christchurch, NZ, 1971).

[3] Barut, A.O. and Bracken, A.J., "Zitterbewegung and the internal geometry of the electron," *Phys. Rev. D* **23** (1981), 2454–2463.

[4] Mack, G. and Salam, A., "Finite-component field representations of the conformal group," *Ann. Phys.* **53** (1969), 174–202.

[5] Bracken, A.J., "$O(4,2)$: an exact invariance algebra for the electron," *J. Phys. A* **8** (1975), 808–815.

[6] Conrad, J.M., "A review of experimental results on neutrino oscillations," *Nucl. Phys. A* **663** (2000), 210C–271C.

[7] Mack, G. and Todorov, I.T., "Irreducibility of the ladder representations of $U(2,2)$ when restricted to the Poincaré group," *J. Math. Phys.* **10** (1969), 2078–2085.

[8] Joos, H., "Theory of representations of the inhomogeneous Lorentz group as a basis for quantum mechanical kinematics," *Forts. der Physik* **10** (1962), 65-146.

[9] Barut, A.O. and Bohm, A., "Reduction of a class of $O(4,2)$ representations with respect to $SO(4,1)$ and $SO(3,2)$," *J. Math. Phys.* **11** (1970), 2938–2945.